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UNIQUE FEATURES OF THE AERODYNAMICS PROBLEMS FOR A WING OF FINITE SPAN

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In linear formulation, the problem of passage of a nonviscous supersonic flow over a thin slightly curved wing of finite span reduces to solution of a wave equation for the velocity potential with time-oriented data on the base plane and the main surface characteristic. Use of Volterra representation for the wave equation solution in this case of specifying initial data then allows us to choose as the defining parameter in the base plane either the exit derivative-wing geometry (direct aerodynamics problem) or the wing load function (converse aerodynamics problem). In addition, the Volterra expression establishes a relationship between the solutions of these problems, eliminating the ambiguity of the converse problem solution [1].

The velocity potential in the direct and converse problems can be written in the form of double integrals, the integrands of which contain singularities. When finding the flow gas dynamics parameters (derivatives of the velocity potential) the degree of the integrand singularities increases, so that at times it is impossible to perform formal differentiation operations within the framework of finite functions, and sometimes differentiation leads to the appearance of singularities for which the integrals become divergent. Often the approach of recognizing the existence of the integrals in the Adamar sense [2] is used. Introduction of such symbolism causes not only complications in realizing solution algorithms, but sometimes demands justification of physically absurd results. In wing theory problems this means acceptance of discontinuities in the gas dynamic parameters upon transition from the disturbed region to the wing surface. However for exact consideration of the character of the integrand singularities, as well as imposition of certain smoothness conditions on the problem's defining parameters, it is possible to represent the flow gas dynamic parameters within the class of finite functions on the wing surface itself. In this study that principal proposition will be illustrated with solutions of the direct and converse aerodynamics problems for a wing with completely supersonic leading edges. In general, the problems of flow over a finite span wing with partially infrasonic leading edges can be formulated immediately in terms of direct and converse aerodynamics problems, and the problems of operations with integrals remains. Moreover, it is necessary to solve singular integral equations. Exact solutions for problems of flow over finite span wings with infrasonic edges are known, reducing to an integral equation with Abel integrand [1, 3, 4], including a solution in the class of finite function [5].

1. For passage of an ultrasonic flow ($M > 1$) over bodies with a spatial configuration which only slightly perturbs the incident flow, the gas dynamics equations can be reduced to a wave equation for the velocity potential [1]

$$\Phi_{xx} - \Phi_{yy} - \Phi_{zz} = 0, \tag{1.1}$$

where the direction of the x-axis of a rectangular Cartesian coordinate system, fixed to the body, coincides with the flow direction at infinity (Fig. 1).

We will consider problems of flow over a thin slightly curved wing, the mean surface of which differs little from some plane parallel to the incident flow velocity. We will term this plane the base plane and transfer boundary conditions from the wing surface to that base plane $S(\eta = 0)$. The region disturbed by the wing is located within the surface Γ_0, which is the envelope of characteristic cones with apices on the supersonic portion of the wing's leading edge. The surface limited by the region of dependence of the point $M(x, y, z)$ above the wing consists of a portion of the surface of the characteristic cone Γ with apex at the point M before its intersection with the main characteristic surface Γ_0 and the base plane S with the portions of Γ_0 and S cut by the cone Γ.

Using the Ostrogradskii–Gauss formula, we can obtain a representation of the solution of wave equation (1.1) in Volterra form [1.6]. Then if for the known solution of the wave equation we choose the Volterra function

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$$w = \int \varphi d\xi = \ln\{[(x-\xi) - r]/\sqrt{(y-\eta)^2 + (z-\zeta)^2}\},$$
$$\varphi = 1/r,\ r = \sqrt{(x-\xi)^2 - [(y-\eta)^2 + (z-\zeta)^2]}, \tag{1.2}$$

then in the Ostrogradskii–Gauss expression the integral over the surface Γ vanishes. And if we assume that on the main characteristic surface $\Phi|_{\Gamma 0} = 0$, the Volterra expression takes on the form

$$\Phi(M) = \frac{1}{2\pi}\frac{\partial}{\partial x}\iint_s (w\Phi'_n - \Phi w'_N)ds. \tag{1.3}$$

Here s is the region of dependence of the point M on the base plane $S(\eta = 0)$;

$$w'_N\Big|_{\eta=0} = \frac{\partial w}{\partial \eta}\Big|_{\eta=0} = -\int \frac{\partial \varphi}{\partial y} d\xi\Big|_{\eta=0} = -\frac{y(x-\zeta)}{r[(z-\zeta)^2 + y^2]} \tag{1.4}$$

is the Volterra conormal derivative function; then with accuracy to second order terms in Eq. (1.3) the conormal derivative $\Phi'_n|_S$ can be replaced by the normal derivative $\Phi'_n|_S$.

Equation (1.3) represents the perturbation potential in terms of the value of the potential $\Phi|_S$ and the normal derivative $\Phi'_n|_S$ on the base plane $S(\eta = 0)$ of a time-oriented type. On the time-oriented surface a dependence exists between the potential Φ and its normal derivative Φ'_n, defined in the general case of a cylindrical base surface by an integrodifferential relationship which can be obtained from the Volterra expression upon the limiting transition of the point M to the base surface. In the current case of flow over a thin wing, where the base surface is the plane $\eta = 0$, according to Eqs. (1.2), (1.4) we have the equalities

$$w(x, y, z)|_{\eta=0} = w(x, -y, z)|_{\eta=0},\ w'_N(x, y, z)|_{\eta=0} = -w'_N(x, -y, z)|_{\eta=0}. \tag{1.5}$$

With consideration of Eq. (1.5) by the method of compensating singularities [1] we can exclude from Eq. (1.3) either the term dependent on the potential value in the plane $\eta = 0$, thereupon representing the potential at the point M(x, y, z) in terms of the normal derivative of the velocity potential:

$$\Phi_1(M) = \frac{1}{\pi}\frac{\partial}{\partial x}\iint_s \Phi'_\eta \int \varphi d\xi ds, \tag{1.6}$$

or the term dependent on the normal derivative of the potential in the plane $\eta = 0$, thereupon writing the potential at M(x, y, z) in terms of the velocity potential itself:

$$\Phi_2(M) = \frac{1}{\pi}\frac{\partial}{\partial x}\iint_s \Phi \int \frac{\partial \varphi}{\partial y} d\xi ds. \tag{1.7}$$

The solutions of the wing overflow problems, Eqs. (1.6), (1.7) have been given in the form of integrodifferential operators. To differentiate the integral operators with variable limits with respect to x, we must consider singularities of the integrands. Thus, in Eq. (1.7), due to a singularity on the line r = 0, direct differentiation is impossible, and to reduce the degree of the singularity we must perform a preliminary integration over the variable ξ, which requires imposition of an additional condition on the smoothness of the defining parameter Φ, i.e., existence of the derivative Φ'_ξ. After differentiating with respect to x, with consideration of the integrand singularities and reduction of the velocity potential on the boundary with the undisturbed region to zero, Eqs. (1.6), (1.7) take on the form [1]

$$\Phi_1 = -\frac{1}{\pi}\iint_s \Phi'_\eta \varphi ds = -\frac{1}{\pi}\iint_s \frac{\Phi'_\eta(\xi, \zeta)}{r} ds; \tag{1.8}$$

$$\Phi_2 = \frac{1}{\pi}\iint_s \Phi'_\xi \int \frac{\partial \varphi}{\partial y} d\xi ds = \frac{y}{\pi}\iint_s \frac{\Phi'_\xi(\xi, \zeta)(x-\xi)}{r[(z-\zeta)^2 + y^2]} ds. \tag{1.9}$$

The potential Φ_1 defines the flow field in the perturbed region in terms of the value of the derivative Φ'_η in the base plane and is the solution of the direct problem. The potential Φ_2 defines the flow field in terms of the derivative Φ'_ξ in the base

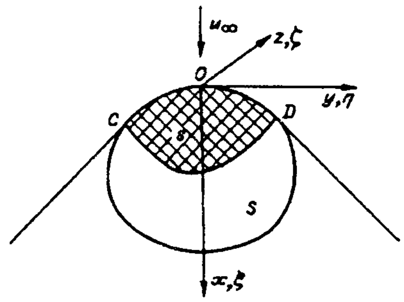

Fig. 1

plane. A linearized Bernoulli equation gives the following relationship between the derivative Φ'_ξ and the pressure drop in the base plane:

$$\Phi'_\xi|_{\eta=0} = \Delta p, \quad \Delta p = [p|_{\eta=-0} - p|_{\eta=+0}]/2\rho_\infty u_\infty.$$

Here u_∞, ρ_∞ are the velocity and density of the undisturbed flow. Equation (1.9) shows the solution of the inverse point — the geometric calculation. According to Eqs. (1.3), (1.6), (1.7) $\Phi = (\Phi_1 + \Phi_2)/2$. And since the disturbance consists of one and the same object, the wing, then the potentials Φ_1, Φ_2 define the same field in different terms: $\Phi = \Phi_1 = \Phi_2$. The potential Φ_2 of the converse problem corresponds to the potential Φ_1 of the direct problem, i.e., the wing configuration defining the flow Φ_1.

In general, the solution of the converse problem is ambiguous. For example, on the base surface $\eta = 0$ to which conditions on the wing surface are transferred, let there be singularities $\int (\partial\varphi/\partial y)d\xi$ (the integrand of the already known potential Φ_2) and $\int (\partial\varphi/\partial x)d\eta$ (integrand of the potential Φ_3) with intensities $c_2(\xi, \zeta)$, $c_3(\xi, \zeta)$ respectively. The potentials of the disturbances created by these singularities will be given by the expressions

$$\Phi_2 = y \iint_s c_2(\xi, \zeta) \frac{(x - \xi)}{r[(z - \zeta)^2 + y^2]} ds; \tag{1.10}$$

$$\Phi_3 = y \iint_s c_3(\xi, \zeta) \frac{(x - \xi)}{r[(x - \xi)^2 - y^2]} ds, \tag{1.11}$$

where for case of calculation, without limiting generality we set the integration limits for the region of dependence $s \in \eta = 0$ of the point M(x, y, z) are limited by the line $\xi = 0$:

$$0 \leq \xi \leq x - y, \quad z - \sqrt{(x - \xi)^2 - y^2} \leq \zeta \leq z + \sqrt{(x - \xi)^2 - y^2}.$$

On the plane $\eta = 0$ we introduce a new coordinate system, [3] $\xi = \xi$, $\zeta = z - \sqrt{(x - \xi)^2 - y^2}\cos\theta$, and rewrite Eqs. (1.10), (1.11) in the form

$$\Phi_2 = y \int_0^{x-y} \int_0^{\pi} \frac{c_2(\xi, z - \sqrt{(x - \xi)^2 - y^2}\cos\theta)(x - \xi)}{(x - \xi)^2\cos^2\theta + y^2\sin^2\theta} d\theta d\xi; \tag{1.12}$$

$$\Phi_3 = y \int_0^{x-y} \int_0^{\pi} \frac{c_3(\xi, z - \sqrt{(x - \xi)^2 - y^2}\cos\theta)(x - \xi)}{(x - \xi)^2\sin^2\theta + y^2\cos^2\theta} d\theta d\xi. \tag{1.13}$$

If we differentiate potentials (1.12), (1.13) with respect to x and then transform to the limit as $y \to 0$, then from the limiting expression we can establish the gas dynamic meaning of the singularities: $c_2(\xi, \zeta) = c_3(\xi, \zeta) = (1/\pi)\Phi'_\xi|_{\eta=0}$. The potential Φ_3, just like the potential Φ_2, is defined in terms of the load $\Phi'_\xi|_{\eta=0}$, but for an identically specified load upon the wing the flow parameters calculated with the potentials Φ_2 and Φ_3 will differ, including the wing geometry.

Commencing from the equivalence of the potentials for direct and converse problems ($\Phi_1 = \Phi_2$) we can establish a relationship between the derivatives Φ'_η, Φ'_ξ in the base plane $\eta = 0$.

2. To find the gas dynamic flow parameters (velocities, pressures) from the expressions for Φ_1, Φ_2 we must differentiate the integral operators with respect to the variables x, y, z, while the integrands contain singularities of several varieties which prohibit direct differentiation. To eliminate these integrand singularities use has been made [1, 3] of the technique of preliminary integration by parts over one of the variables in the double integral operators, which requires imposition of additional conditions upon the smoothness of the defining parameters on the wing surface. Sometimes [2], while

performing differentiation operations incorrect within the framework of finite functions, the concept of the finite portion of the diverging integral has been introduced, making it possible to formally carry out the required operations, yet producing results the physical essence of which is absurd. Most often the appearance of the diverging integral itself is either a result of erroneous estimation of certain terms in the limiting transitions, or of neglect of a transition rule in the singular integrals dependent upon the parameter. In fact, the expressions for the potentials Φ_1, Φ_2 and the expressions for the gas dynamic flow parameters obtained by differentiation of Φ_1, Φ_2 are written in terms of the defining parameters on the wing surface in the disturbed space around the wing, outside the wing surface itself. Upon the limiting transition to the wing surface ($y \to 0$) it must be recalled that the rule

$$\lim_{\varepsilon \to 0} \int f(x, \varepsilon)dx = \int f(x, 0)dx. \tag{2.0}$$

is not always satisfied for integrals dependent upon a parameter. Uncontrolled use of this rule for integrals, the integrands of which contain singularities, leads to the appearance of divergent integrals. Observation of the rules for differentiation of integrals with variable limits and the requirement of necessary smoothness of the defining parameters on the wings permit acquisition of flow gas dynamic parameter values within the class of finite function on the very wing surface ($y \to 0$) as well.

We will first demonstrate the invalidity of the widely used (see, for example, [2]) expression

$$\Phi'_y \big|_{y=0} = -\Phi'_x \big|_{y=0} + \frac{1}{\pi} \iint\limits_s \frac{\Phi'_\xi(\xi, \zeta)\,(x - \xi)}{(z - \zeta)^2 \sqrt{(x - \xi)^2 - (z - \zeta)^2}}\, d\zeta d\xi, \tag{2.1}$$

establishing a relationship between Φ'_y and Φ'_x in the base plane $y = 0$, commencing from solution of the converse problem Φ_2 (see Eq. (1.9)). The integral standing on the right side of Eq. (2.1) diverges, but it has been suggested that it be treated in the Adamar sense. The appearance in Eq. (2.1) of a diverging integral is the result of departure from the rules of differentiation for singular integrals with variable limits, which then spawns false estimates of the terms dropped in the limiting transition.

We write the potential Φ_2 (1.12) in the form

$$\Phi_2 = \frac{y}{\pi} \int_0^{x-y} \int_0^{\pi} \frac{\Phi'_\xi(\xi, z - \sqrt{(x - \xi)^2 - y^2}\cos\theta)\,(x - \xi)}{(x - \xi)^2 \cos^2\theta + y^2 \sin^2\theta}\, d\theta d\xi. \tag{2.2}$$

As was said above, without loss of generality as to the character of integral singularities, for the boundary of the perturbed region, we will take the line $\xi = 0$.

Writing the potential Φ_2 in the form of Eq. (2.2) allows direct differentiation with respect to y (and x as well). We will differentiate in sequence with respect to y, the cofactor y appearing before the integral, the integrand (first the core, then the cofactor $\Phi'_\xi(\xi, z - \sqrt{(x - \xi)^2 - y^2}\cos\theta)$):

$$\begin{aligned}
\Phi'_{2y} = &-\frac{y^2}{\pi} \int_0^{\pi} \frac{\Phi'_\xi(x - y, z)d\theta}{y^2 \cos^2\theta + y^2 \sin^2\theta} \\
&+ \frac{1}{\pi} \int_0^{x-y} \int_0^{\pi} \frac{\Phi'_\xi(\xi, z - \sqrt{(x - \xi)^2 - y^2}\cos\theta)\,(x - \xi)}{(x - \xi)^2 \cos^2\theta + y^2 \sin^2\theta}\, d\theta d\xi \\
&- \frac{2y^2}{\pi} \int_0^{x-y} \int_0^{\pi} \frac{\Phi'_\xi(\xi, z - \sqrt{(x - \xi)^2 - y^2}\cos\theta)\,(x - \xi)\sin^2\theta}{[(x - \xi)^2 \cos^2\theta + y^2 \sin^2\theta]^2}\, d\theta d\xi \\
&+ \frac{y^2}{\pi} \int_0^{x-y} \int_0^{\pi} \frac{\Phi''_{\xi\zeta}(\xi, z - \sqrt{(x - \xi)^2 - y^2}\cos\theta)\,(x - \xi)\cos\theta}{\sqrt{(x - \xi)^2 - y^2}\,[(x - \xi)^2 \cos^2\theta + y^2 \sin^2\theta]}\, d\theta d\xi.
\end{aligned} \tag{2.3}$$

Transforming in Eq. (2.3) from the integration variables ξ, θ to the variables ξ, ζ, we have

$$\Phi'_y(x, y, z) = -\Phi'_\xi(x-y, z) + \frac{1}{\pi}\iint_s \frac{\Phi'_\xi(\xi,\zeta)(x-\xi)}{r[(z-\zeta)^2+y^2]}\,d\zeta d\xi$$
$$-\frac{2y^2}{\pi}\iint_s \frac{\Phi'_\xi(\xi,\zeta)(x-\xi)r}{[(z-\zeta)^2+y^2]^2[(x-\xi)^2-y^2]}\,d\zeta d\xi \tag{2.4}$$
$$+\frac{y^2}{\pi}\iint_s \frac{\Phi''_{\xi\zeta}(\xi,\zeta)(x-\xi)(z-\zeta)}{r[(z-\zeta)^2+y^2][(x-\xi)^2-y^2]}\,d\zeta d\xi,$$

where the limits of the region of s dependence are as follows:

$$0 \le \xi \le x-y,\quad z-\sqrt{(x-\xi)^2-y^2} \le \zeta \le z+\sqrt{(x-\xi)^2-y^2}.$$

If in Eq. (2.4) we immediately take $y = 0$, we obtain Eq. (2.1) (derived by another manner in [2]). However the integral operators with cofactor y^2 dropped in Eq. (2.4) have singularities of such high order that not only do they do not vanish as $y \to 0$, but remain comparable to the divergent integral in Eq. (2.1).

To simplify establishing estimates, we will initially assume $\Phi'_\xi(\xi, \zeta) = f_0(\xi)$ and carry out integration in Eq. (2.4). In consideration of the significance of this step we will describe the integration process in greater detail.

The first integral operator is

$$J_1 = \frac{1}{\pi}\int_0^{x-y} f_0(\xi)(x-\xi)J_1^* d\xi,\quad J_1^* = \int_{z-\sqrt{(x-\xi)^2-y^2}}^{z+\sqrt{(x-\xi)^2-y^2}} \frac{d\zeta}{r[(z-\zeta)^2+y^2]} = \frac{\pi}{y(x-\xi)}, \tag{2.5}$$
$$J_1 = \frac{1}{y}\int_0^{x-y} f_0(\xi)d\xi.$$

The second integral operator is

$$J_2 = -\frac{2y^2}{\pi}\int_0^{x-y} \frac{f_0(\xi)(x-\xi)}{[(x-\xi)^2-y^2]} J_2^* d\xi,\quad J_2^*$$
$$= \int_{z-\sqrt{(x-\xi)^2-y^2}}^{z+\sqrt{(x-\xi)^2-y^2}} \frac{r\,d\zeta}{[(z-\zeta)^2+y^2]^2} = \frac{[(x-\xi)^2-y^2]}{2y^2} J_1^*, \tag{2.6}$$
$$J_2 = -\frac{1}{y}\int_0^{x-y} f_0(\xi)d\xi.$$

The third integral operator J_3, in light of the fact that we consider the case of constant loading over the extent $\Phi'_\xi(\xi, \zeta) = f_0(\xi)$ and $\Phi''_{\xi,\zeta}(\xi, \zeta) = 0$, is equal to zero.

The first integral operator J_1, corresponding to the divergent integral from Eq. (2.1), tends to infinity as $y \to 0$, but is compensated by the operator J_2, $J_1 + J_2 = 0$.

Thus, according to Eq. (2.4), for $\Phi'_\xi(\xi, \zeta) = f_0(\xi)$ and the leading edge of the dependence range $\xi = 0$

$$\Phi'_y(x, y, z) = -\Phi'_x(x-y, 0, z),\ \Phi'_y(x, y) = -\Phi'_x(x-y, 0),$$
$$\Phi'_y(x, 0) = -\Phi'_x(x, 0),$$

which then corresponds to a planoparallel flow pattern.

According to Eq. (2.1), in the case under consideration

$$\Phi'_y(x, 0, z) = -\Phi'_x(x, 0, z) + \frac{1}{\pi}\int_0^x f_0(\xi)(x-\xi)J_0^* d\xi,\quad J_0^* = \int_{z-(x-\xi)}^{z+(x-\xi)} \frac{d\zeta}{(z-\zeta)^2\sqrt{(x-\xi)^2-(z-\zeta)^2}} = 2\frac{\sqrt{(x-\xi)^2-\varepsilon^2}}{\varepsilon(x-\xi)^2}.$$

With an accuracy to ε^2, where ε is the interval of the singularity in question ($z - \varepsilon \le \zeta \le z + \varepsilon$), we have $J_0 = 2/\varepsilon(x - \xi)$. Then

$$\Phi'_y(x, 0, z) = -\Phi'_x(x, 0, z) + \frac{2}{\varepsilon\pi}\int_0^x f_0(\xi)d\xi.$$

The first term on the right coincides with the first term of Eq. (2.4), and the second term tending to infinity as $\varepsilon \to 0$ "cries" to be dropped. This is not a compromise of linear theory, but the result of incorrect operations with the integrals.

In the case of a linear law of loading change along the extent of the wing ($\Phi'_\xi(\xi, \zeta) = f_1(\xi)\zeta$) the relationship between the first and second integral operators of Eq. (2.4) remains as before.

The first integral operator is

$$J_1 = \frac{1}{\pi}\int_0^{x-y} f_1(\xi)(x - \xi)J^*_{11}d\xi, \quad J^*_{11} = \int_{z-\sqrt{(x-\xi)^2-y^2}}^{z+\sqrt{(x-\xi)^2-y^2}} \frac{\zeta d\zeta}{r[(z - \zeta)^2 + y^2]} = zJ^*_1$$

(J_1^* is defined in Eq. (2.5)). Then

$$J_1 = \frac{z}{y}\int_0^{x-y} f_1(\xi)d\xi.$$

The second integral operator is

$$J_2 = -\frac{2y^2}{\pi}\int_0^{x-y} \frac{f_1(\xi)(x - \xi)}{[(x - \xi)^2 - y^2]} J^*_{21}d\xi, \quad J^*_{21} = \int_{z-\sqrt{(x-\xi)^2-y^2}}^{z+\sqrt{(x-\xi)^2-y^2}} \frac{\zeta r d\zeta}{[(z - \zeta)^2 + y^2]} = zJ^*_2$$

(J_2^* is defined in Eq. (2.6)). Then

$$J_2 = -\frac{z}{y}\int_0^{x-y} f_1(\xi)d\xi.$$

The third integral operator for $\Phi''_{\xi,\zeta}(\xi, \zeta) = f_1(\xi)$ has the form

$$J_3 = \frac{y^2}{\pi}\int_0^{x-y} \frac{f_1(\xi)(x-\xi)}{[(x - \xi)^2 - y^2]} J^*_{31}d\xi, \quad J^*_{31} = \int_{z-\sqrt{(x-\xi)^2-y^2}}^{z+\sqrt{(x-\xi)^2-y^2}} \frac{(z - \zeta)d\zeta}{r[(z - \zeta)^2 + y^2]} = 0,$$

consequently, $J_3 = 0$.

In the case being considered of linear change in loading along the wing the operators J_1, J_2 also compensate each other: $J_1 + J_2 = 0$. Equation (2.4) gives the following relationship between the geometry and loading on the wing: $\Phi'_y(x, 0, z) = -\Phi'_x(x, 0, z)$. The local relationship can be explained by the antisymmetry of the load value relative to the plane $\zeta = z$.

According to Eq. (2.1) for $\Phi'_\xi(\xi, \zeta) = f_1(\xi)\zeta$ we obtain

$$\Phi'_y(x, 0, z) = -\Phi'_x(x, 0, z) + \frac{2z}{\varepsilon\pi}\int_0^x f_1(\xi)d\xi.$$

The law of loading change can be written in the quite general form $\Phi'_\xi(\xi, \zeta) = \sum_{i=0}^{n} f_i(\xi)\zeta^i$ or in a form convenient for evaluating the singularities of the integral operators:

$$\Phi'_\xi(\xi, \zeta) = \sum_{i=0}^{n} f_i(\xi)[z - (z - \zeta)]^i = \sum_{i=0}^{n} g_i(\xi)(z - \zeta)^i.$$

The behavior of the integral operators for $i = 0, 1$ was considered above. For $i \ge 2$ there appear in the numerators of the integral operators of Eq. (2.4) yet other terms of the form $(z - \zeta)^i$, which reduce the degree of integrand singularity $[(z - \zeta)^2 + y^2]^{-1}$, $[(z - \zeta)^2 + y^2]^{-2}$ as $y \to 0$, the integral operators becoming regular.

Thus, the relationship between the derivatives Φ'_y, Φ'_x, calculated on the basis of the potential Φ_2 is representable within the class of finite functions both within the flow and in the limiting transition to the wing.

The possibility of obtaining a regular solution to the converse problem was analyzed by representing the potential Φ_2 in the form of Eq. (2.2) to establish a relationship to the incorrect solution (2.1) from [2]. But, as has already been noted at the start of section 2, the rational method of finding derivatives of the potentials expressed in terms of singular integrals is reduction of the degree of singularity by preliminary integration by parts over one of the variables. This then permits differentiation of the integrals with variable limits without introducing any arbitrary differentiation symbolism which leads to nonphysical results.

In [1] the integral operators for the potentials Φ_1, Φ_2 were subjected to preliminary integration by parts over the variable ζ as well as the variable ξ, making it possible to differentiate the potential Φ_1 (solution of the direct problem) with respect to x or to differentiate the potential Φ_2 (solution of the converse problem) with respect to y. Meanwhile in performing the preliminary integration by parts it is necessary to impose additional conditions on the defining parameters $\Phi'_\eta|_{\eta=0}$, $\Phi'_\xi|_{\eta=0}$ the existence of the corresponding second derivatives $\Phi''_{\eta\zeta}$, $\Phi''_{\xi\zeta}$, $\Phi''_{\eta\xi}$, $\Phi''_{\xi\xi}$.

The derivatives Φ'_{1x}, defined in the direct problem have the form

$$\Phi'_{1x}(x, y, z) = -\Phi'_y(x - y, 0, z) - \frac{1}{\pi}\int_{COD}\left[\Phi'_\eta \int \frac{\partial\varphi}{\partial x}\, d\zeta\right]_{\zeta=f(\xi)} d\xi \tag{2.7}$$

$$+ \frac{1}{\pi}\iint_s \Phi''_{\eta\zeta} \int \frac{\partial\varphi}{\partial x}\, d\zeta d\zeta d\xi;$$

$$\Phi'_{1x}(x, y, z) = -\frac{1}{\pi}\int_{COD} [\Phi'_\eta \varphi]_{\xi=\psi(\zeta)} d\zeta - \frac{1}{\pi}\iint_s \Phi''_{\eta\xi}\varphi\, d\xi d\zeta, \tag{2.8}$$

while the derivatives Φ'_{2y}, defined in the converse problem appear as:

$$\Phi'_{2y}(x, y, z) = -\Phi'_x(x - y, 0, z) + \frac{1}{\pi}\int_{COD}\left[\Phi'_\xi \iint \frac{\partial^2\varphi}{\partial y^2}\, d\xi d\zeta\right]_{\zeta=f(\xi)} d\xi \tag{2.9}$$

$$-\frac{1}{\pi}\iint_s \Phi''_{\xi\zeta} \iint \frac{\partial^2\varphi}{\partial y^2}\, d\xi d\zeta d\zeta d\xi;$$

$$\Phi'_{2y}(x, y, z) = -\frac{1}{\pi}\int \left[\Phi'_\xi \iint \frac{\partial^2\varphi}{\partial y^2}\, d\xi d\xi\right]_{\xi=\psi(\zeta)} d\zeta \tag{2.10}$$

$$-\frac{1}{\pi}\iint_s \Phi''_{\xi\xi} \iint \frac{\partial^2\varphi}{\partial y^2} d\xi d\xi d\xi d\zeta.$$

Here s is the region of dependence of the point M(x, y, z); COD($\zeta = f(\xi)$, $\xi = \psi(\zeta)$) is the boundary between the region of dependence and the unperturbed region (see Fig. 1).

In expanded form the integrand of the operators in Eqs. (2.7)-(2.10) can be written as follows:

$$\int \frac{\partial\varphi}{\partial x}\, d\zeta = \frac{(x - \xi)(z - \zeta)}{r[(x - \xi)^2 - y^2]}; \tag{2.11}$$

$$\varphi = r^{-1}; \tag{2.12}$$

$$\iint \frac{\partial^2\varphi}{\partial y^2}\, d\xi d\zeta = \frac{(x - \xi)(z - \zeta)(r^2 - y^2)}{r[(z - \zeta)^2 + y^2][(x - \xi)^2 - y^2]}; \tag{2.13}$$

$$\iint \frac{\partial^2\varphi}{\partial y^2}\, d\xi d\xi = \frac{y^2(x - \xi)^2 - r^2(z - \zeta)^2}{r[(z - \zeta)^2 + y^2]^2}. \tag{2.14}$$

All these integrands have a singularity at r = 0. Upon transition of the point M(x, y, z) to the base plane (y = 0) the line ζ = z and the point ξ = x become singular; in some cases the singularity is such that the integral is divergent. But this does not imply the absence of a solution in the ordinary meaning of that term. To prove the existence of a solution in this case, when calculating the integrals we must perform the limiting transition $y \to 0$ after carrying out the integration. The integrands reflect the character of the effect of the hydrodynamic singularities (turbulent sections, sources, disturbances of more complex character, etc.) on the point M(x, y, z) in the space outside the plane where the hydrodynamic singularities are located. Each hydrodynamic singularity has its own permissible path for transition of the point M(x, y, z) into the singularity plane. For certain singularities (sources) the order of the transition of point M into the perturbing plane and the order of integration in the double integral make no difference. For other types of singularities (turbulent sections) the result depends upon the order of integration of the double integral, while in the case of more complex disturbances it may also depend on the order of the limiting transition $y \to 0$, i.e., before or after integration.

Without loss of generality we will analyze the singularities of the integral operators in Eqs. (2.7)-(2.10) for a wing where the boundary COD of the dependence region with the undisturbed flow region is given by the equation $\xi = 0$ (Fig. 1).

According to Eq. (2.11) the integral operator from Eq. (2.7) in which the hydrodynamic singularity $\int (\partial\varphi/\partial x)d\zeta$ is related to a turbulent segment in the direction of the ζ-axis can be described at $y = 0$ in the following manner:

$$J_7(x, 0, z) = \int\limits_0^x \int\limits_{z-(x-\xi)}^{z+(x-\xi)} \Phi''_{\pi\zeta} \frac{(z-\zeta)}{r(x-\xi)} d\zeta d\xi.$$

We will write the law for change in slope of the normal in the form of a series

$$\Phi''_{\pi\zeta}(\xi, \zeta) = \sum_{i=0}^{n} f_i(\xi)\zeta^i \quad \text{или} \quad \Phi''_{\pi\zeta}(\xi, \zeta) = \sum_{i=0}^{n} g_i(\xi)(z-\zeta)^i$$

(when series in ζ are used to express the functions $\Phi''_{\eta\zeta}$, we use one and the same notation for the coefficients: $f_i(\xi)$, $g_i(\xi)$). Then at $i = 0$

$$J_{70}(x, 0, z) = \int\limits_0^x \frac{f_0(\xi)}{(x-\xi)} \int\limits_{z-(x-\xi)}^{z+(x-\xi)} \frac{(z-\zeta)}{r} d\zeta d\xi = 0;$$

while at $i = 1$

$$J_{71}(x, 0, z) = \int\limits_0^x \frac{f_1(\xi)}{(x-\xi)} \int\limits_{z-(x-\xi)}^{z+(x-\xi)} \frac{[z-(z-\zeta)](z-\zeta)}{r} d\zeta d\xi = \frac{\pi}{2}\int\limits_0^x f_1(\xi)(x-\xi)d\xi.$$

For $i \geq 2$ there appear in the inner integral over ζ terms with odd and even powers of the factor $(z - \zeta)$. Integrals with terms having an odd power of $(z - \zeta)^{2i+1}$ vanish, while after finding the integrand, terms with $(z - \zeta)^{2i+2}$ vanish, while after finding the integrand, terms with $\pi(x - \xi)^2$, have a cofactor $(x - \xi)^{-1}$ so that the singularity ξ in the external integral vanishes and all J_{7i} are proportional to the integral $\pi \int\limits_0^x f_i(\xi)(x - \xi)d\xi$. It can be shown that condition (2.0) is satisfied for J_7.

The integral operator from Eq. (2.8), in which the hydrodynamic singularity r^{-1} is a supersonic source, has an integrable singularity, allowing change in the order of integration and use of rule (2.0).

The integrand of the operator from Eq. (2.9), in light of the fact that $\varphi_{xx} - \varphi_{yy} - \varphi_{zz} = 0$ and Eq. (2.13) can be written as

$$\begin{aligned}\iint \frac{\partial^2\varphi}{\partial y^2} d\xi d\zeta &= \int \frac{\partial\varphi}{\partial z} d\xi - \int \frac{\partial\varphi}{\partial x} d\zeta = \\ &= \frac{(x-\xi)(z-\zeta)}{r[(z-\zeta)^2 + y^2]} - \frac{(x-\xi)(z-\zeta)}{r[(x-\xi)^2 - y^2]},\end{aligned} \tag{2.15}$$

and is related to turbulent segments in the directions ξ and ζ. Singularities in the integral operator with integrand $\int (\partial\varphi/\partial x)d\zeta$ were analyzed above. A portion of the integral operator with integrand $\int (\partial\varphi/\partial z)d\xi$ at $y = 0$ according to Eqs. (2.9), (2.11). (2.13), (2.15) has the form

$$J_9(x, 0, z) = \int\limits_0^x \int\limits_{z-(x-\xi)}^{z+(x-\xi)} \Phi''_{\xi\xi} \frac{(x-\xi)}{r(z-\zeta)} d\zeta d\xi.$$

We write $\Phi''_{\xi\zeta}(\xi\ \zeta) = \sum_{i=0}^{n} f_i(\xi)\zeta^i$. Then at i = 0.

$$J_{90}(x, 0, z) = \int_0^x f_0(\xi)(x - \xi) \int_{z-(x-\xi)}^{z+(x-\xi)} \frac{d\zeta}{r(z - \zeta)} d\xi = 0;$$

at i = 1,

$$J_{91}(x, 0, z) = \int_0^x f_1(\xi)(x - \xi) \int_{z-(x-\xi)}^{z+(x-\xi)} \frac{[z - (z - \zeta)]}{r(z - \zeta)} d\zeta d\xi = -\pi \int_0^x f_1(\xi)(x - \xi) d\xi;$$

for i ≥ 2 the inner integrals over ζ, aside from the terms

$$\int_{z-(x-\xi)}^{z+(x-\xi)} \frac{d\zeta}{r(z - \zeta)} = 0, \quad \int_{z-(x-\xi)}^{z+(x-\xi)} \frac{d\zeta}{r} = \pi,$$

will still have integrable terms of the type $\int [(z - \zeta)^i/r]d\zeta$. As for J_7, rule (2.0) is valid for J_9.

The integral operator of Eq. (2.10) with integrand (2.14) for y = 0 on the line $\zeta = z$ has a nonintegrable singularity $(z - \zeta)^{-2}$:

$$J_{10}(x, 0, z) = \int_{z-x}^{z+x} \frac{A(\zeta)}{(z - \zeta)^2} d\zeta, \quad A(\zeta) = \int_0^{x-(z-\zeta)} \Phi''_{\xi\xi} r d\xi, \tag{2.16}$$

which implies a nonphysical result—finite change in loading corresponds to infinite deformation of the wing surface. The integrand of the operator in Eq. (2.16) can be obtained from the integrand of the operator

$$J_{10}(x, y, z) = \int_{z-\sqrt{x^2-y^2}}^{z+\sqrt{x^2-y^2}} \int_0^{x-\sqrt{(z-\zeta)^2+y^2}} \Phi''_{\xi\xi} \frac{y^2(x - \xi)^2 - r^2(z - \zeta)^2}{r[(z - \zeta)^2 + y^2]^2} d\xi d\zeta \tag{2.17}$$

by dropping terms therein with the cofactor y^2 as $y \to 0$.

Dropping of terms with y^2 as $y \to 0$ before integration is carried out is impermissible since the term in the integrand of Eq. (2.17) with numerator $y^2(x - \xi)^2$ would have a term with cofactor y^{-1}. Such a y^{-1} singularity also develops after integration in the term with numerator $r^2(z - \zeta)^2$, appearing in operator (2.16). But in the given case of integration for $y \neq 0$ the terms with y^{-1} singularities singularity from the numerator terms $[y^2(x - \xi)^2 - r^2(z - \zeta)^2]$ cancel each other. No other singularities appear upon integration, the operator $J_{10}(x, 0, z)$ exists given the condition of finiteness of $\Phi''_{\xi\xi}(\xi, \zeta)$ and $\lim_{y\to 0} J_{10}(x, y, z)$ is a finite value. The procedure for calculating the double integrals from Eq. (2.17) is quite cumbersome, and to prove the existence of $J_{10}(x, y, z)$ it is sufficient to carry out the former for $\Phi''_{\xi\xi}(\xi, \zeta) = \text{const} = c$. In that case $J_{10}(x, y, z) = c\pi x$ over the entire disturbed region, including y = 0.

In Eq. (2.10), which defines $\Phi'_{2y}(x, y, z)$, in terms of load parameters along the wing, aside from the double integral of the operator $J_{10}(x, y, z)$, there is a single integral along the contour COD($\xi = 0$), dependent on $\Phi'_\xi(\xi, \zeta)$:

$$J^1_{10}(x, y, z) = \int_{z-\sqrt{x^2-y^2}}^{z+\sqrt{x^2-y^2}} \Phi'_\xi(0, \zeta) \frac{y^2x^2 - r^2(z - \zeta)^2}{r[(z - \zeta)^2 + y^2]^2} d\zeta. \tag{2.18}$$

Here integration must be performed at $y \neq 0$. After integration terms with the factor y^{-1} appear which cancel each other. The integral $J^1_{10}(x, y, z)$ exists, given finiteness of $\Phi'_\xi(0, \zeta)$. For proof it is sufficient to consider specification of a load on the wing in the form $\Phi'_\xi(\xi, \zeta) = p_0 + c\xi$, then $\Phi'_\xi(0, \zeta)$. The integral of Eq. (2.18) then has the form $J^1_{10}(x, y, z) = p_o\pi$ over the entire disturbed region including y = 0.

Equation (2.10) in the given case of a boundary COD($\xi = 0$) for a load $\Phi'_{\xi} = p_0 + c\xi$ can be written in the form $\Phi'_{2y}(x, y, z) = -(p_0 + cx)$, which corresponds to the physical solution of the problem for a wing of infinite span. For $c = 0$ $\Phi'_{2y}(x, y, z) = -p_0$ is the solution of the problem for a planar plate.

Thus, the Volterra solution makes possible establishment of an unambiguous correspondence between the direct and converse wing aerodynamics problems. The flow gas dynamic parameters in the direct and converse problems are representable in terms of the defining parameters on the wing surface within the class of finite functions, including the transition to the wing surface, thus establishing the relationship between the defining parameters (geometry and loading) on the wing surface.

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